Riemann-Roch + Algebraic Curves

<u>Goals</u>:

1) Review for analysis/topology qual

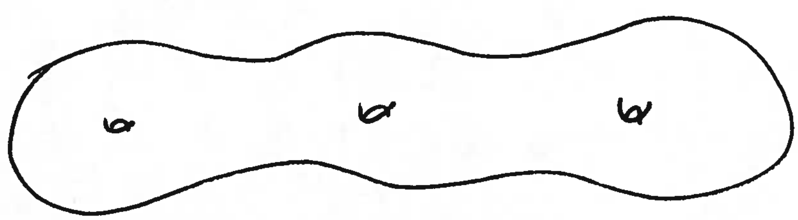
complex analysis ⇝ alg. geometry
Riemann surfaces   Algebraic curves.

2) Motivate divisors + Riemann Roch.

3) State gonality + Brill-Noether Thm.

§1: Complex Analysis

<u>Defn</u>: A Riemann surface X of genus g is a 1-dim. compact manifold



genus 3.

<u>Rmk</u>: Most of what I say will apply to alg curves. i.e. complete, smooth algebraic variety of dim 1., replacing "meromorphic" with "morphism" as needed

<u>Prop</u>: Every non-constant ~~function~~ meromorphic function $g: X \to \mathbb{P}^1(\mathbb{C})$ has a pole.

Pf: If g holomorphic (no pole) & nonconstant, open mapping thm.

$\Rightarrow \mathrm{Im}(g)$ open. $\subseteq \mathbb{C}$

But X cmpt $\Rightarrow \mathrm{Im}(g)$ cmpct. $\subseteq \mathbb{C}$ $\lightning$

Prop: If $f, g : X \to \mathbb{P}^1(\mathbb{C})$ have same zeroes, poles (w/ multiplicities) then $f = \lambda g$.

Pf: $\frac{f}{g}$ has no poles $\Rightarrow \frac{f}{g} = \lambda \in \mathbb{C}$.

~~Prop: # zeroes of f = # poles of f.~~

Prop: $f : X \to \mathbb{P}^1(\mathbb{C})$ non-constant has the same # of zeroes and poles.

Pf: f actually assumes every value a constant # of times given by topological degree

Prop **Complex-Tori** (genus 1 case)

$X = \mathbb{C}/\Lambda$ - not all conformally equivalent!

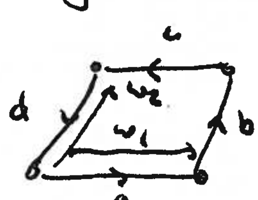
Prop: $f : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$

zeroes $z_1 \dots z_r$ poles $p_1 \dots p_s$

mult. $n_1 \dots n_r$ $m_1 \dots m_s$

then $\sum_{i=1}^{r} z_i n_i = \sum_{j=1}^{s} p_j m_j$. $(\mathrm{mod}\ \Lambda)$.

Pf: Pick fundamental parallelpiped



in $\mathbb{C}$

w/ no zeroes + poles on boundary.

$$\# \text{ zeros} - \# \text{ poles} = \int_{\square} \frac{f'}{f}\, dz \overset{\text{want}}{=} 0$$

$$= \int_a + \int_b + \int_c + \int_d$$

$$\int_c \frac{f'(z)}{f(z)}\, dz \qquad \tilde{z} = z - \omega_2 \qquad dz = d\tilde{z}$$

$$= \int_{-a} \frac{f'(\tilde{z}+\omega_2)}{f(\tilde{z}+\omega_2)}\, d\tilde{z} \overset{\text{periodic}}{=} \int_{-a} \frac{f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}$$

$\Rightarrow$ a, c cancel, b, d cancel.

$$\sum n_i z_i - \sum m_j p_j = \int_{\square} \frac{z f'(z)}{f(z)}\, dz \overset{\text{want}}{\in} \Lambda.$$

$$\int_c \frac{z f'(z)}{f(z)}\, dz \qquad \tilde{z} = z - \omega_2$$

$$= \int_{-a} \frac{(\tilde{z}+\omega_2)\, f'(\tilde{z}+\omega_2)}{f(\tilde{z}+\omega_2)}\, d\tilde{z} \overset{\text{periodic}}{=} \int_{-a} \frac{(\tilde{z}+\omega_2)\, f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}$$

$$= \underbrace{\int_{-a} \frac{\tilde{z}\, f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}}_{\text{cancels w/ } \int_a} + \omega_2 \int_{-a} \frac{f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}$$

set $v = f(\tilde{z})$

$dv = f'(\tilde{z})\, d\tilde{z}$

$$= \omega_2 \int_{f(-a)} \frac{1}{v}\, dv$$

: $f(-a)$ closed loop not through 0 (by periodicity)

$$= \omega_2 \, (\text{winding } \#) \in \Lambda.$$

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Thm: (Abel–Jacobi)

This condition is sufficient.

Rmk: This implies there does not exist $f : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ w/ 1 simple pole.

pf. If p pole, z zero, we'd have $p = z$. This is lunacy !.

Q: How Does this relate to A.G.?

Def: Weierstrass $\wp$-function (for lattice Λ)

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \in \Lambda}}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$\wp'(z) = \frac{-2}{z^3} + \sum_{\omega \neq 0 \in \Lambda}' \frac{-2}{(z-\omega)^3}$$

Ex. $\wp, \wp'$ are doubly-periodic w/ poles exactly on Λ.

Prop:

$$\begin{aligned} f : \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto [\wp(z) : \wp'(z) : 1] = [x : y : z] \\ 0 \ (\mathrm{mod}\ \Lambda) &\longmapsto [0 : 1 : 0] \end{aligned}$$

is an embedding onto a (Zariski)-closed subset.

"Proof:" V = v.s. of functions $\mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ with pole of order ≤ 6 on $0 \pmod{\Lambda}$

Claim: $\dim(V) \leq 6$.

pf: If $g \in V$, take a Laurent expansion

$$g(z) = \frac{c_6}{z^6} + \dots + c_0 + \frac{c_1}{z} + \tilde{g}(z)$$

If h has same (almost) Laurent expansion, $\neq g$.

$$h(z) = \frac{c_6}{z_6} + \dots + c_0 + \frac{b_1}{z} + \tilde{h}(z)$$

$\Rightarrow$ $g-h$ - doubly periodic w/ 1 simple pole, ↯

$\Rightarrow$ $g = h$ (i.e. $c_6, \dots c_2, c_0$ determine g).

Consider these 7 functions

$1, \wp(z), \wp(z)^2, \wp(z)^3, \wp'(z), \wp'(z)^2, \wp'(z)\wp(z)$

$\exists$ linear relation among these

$\Updownarrow$

pt.s in $\mathrm{Im}(f)$ satisfy some polynomial.

Ex. $$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

i.e. pts in $\mathrm{Im}(f)$ satisfy

$$y^2 = 4x^3 - g_2 x - g_3$$

homogenize $\rightsquigarrow$ $$zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3$$

pt. @ infinity - set $z=0 \Rightarrow x=0$, so $y = \lambda \in \mathbb{C}^*$.

Summarize

Want $X \hookrightarrow \mathbb{P}^r(\mathbb{C})$ for some r.

D = some subset of "allowed poles"

$f_1 \dots f_n$ some # of merom. functions w/ enough poles to be inj. in D.

v.s. of functions w/ allowed poles is finite

$\Rightarrow f_1 \dots f_n$ satisfy some polynomials

§2: Divisors & Riemann-Roch

Def. Divisors on X is the free abelian gp. on points of X

$$D = \sum_{p \in X} n_p [p]$$ all but finitely many n_p are zero.

Def. $f : X \to \mathbb{P}^1(\mathbb{C})$, principal divisor assoc. to f

$$(f) = \sum_{p \in X} \mathrm{ord}_p(f)\,[p]$$

Def. $D = \sum n_p [p]$, ~~then~~ the degree of D

~~$\deg(D) = \sum n_p$~~

$$\deg : \text{Divisors} \longrightarrow \mathbb{Z}$$
$$\sum n_p [p] \longmapsto \sum n_p.$$

Rmk: $\deg((f)) = 0$.

Def: D, E are equivalent if
$$D - E = (f)$$

Def: $\mathcal{L}(D) = \{ h : X \to \mathbb{P}^1(\mathbb{C}) \mid \underbrace{(h) + D \geq 0}_{\text{all coeff non-neg.}} \}$

Rmk $f \notin \mathcal{L}((f))$, but $\frac{1}{f} \in \mathcal{L}((f))$

Prop. If $D \sim E$, $\mathcal{L}(D) \cong \mathcal{L}(E)$

Pf: $\cancel{\mathcal{L}(D) \longrightarrow \mathcal{L}(E)}$

$\cancel{g \longmapsto}$

$D - E = (f)$

Consider $\mathcal{L}(D) \xrightarrow{\cong} \mathcal{L}(E)$
$$g \longmapsto f \cdot g$$

w.t.s. $(fg) + E \geq 0$

$$(fg) + E = (f) + (g) + E$$
$$= D - E + (g) + E = (g) + D \geq 0.$$

Def: $\ell(D) = \dim \mathcal{L}(D)$

- well-defined on equiv. classes.

$\deg(D)$ also well-defined

(since $\deg((f)) = 0$)

Def: The canonical divisor

ω is a 1-form on X.

$$(\omega) = \sum_p \text{ord}_p(\omega)\,[p]$$

- This is well-defined

nd. of $U \xrightarrow{f} X$, $f^*(\omega) = g(z)\,dz$.

Prop ω_1, ω_2 1-forms, define a function

$$\frac{\omega_1}{\omega_2}(z) = \frac{g_1(z)}{g_2(z)} \qquad \begin{aligned} f^*(\omega_1) &= g_1(z)\,dz \\ f^*(\omega_2) &= g_2(z)\,dz \end{aligned}$$

- also well-defined (check)

$\Rightarrow (\omega_1) \sim (\omega_2)$

Riemann - Roch

For any divisor D on X, (of genus g)

$$\ell(D) - \ell(K_X - D) = \deg(D) - g + 1$$

Prop i) $\ell(0) = 1$

ii) $\ell(K_X) = g$

iii) $\deg(K_X) = 2g - 2$.

i.) All functions w/ no poles are constant

ii.) $\ell(0) - \ell(K_X - 0) = \deg(0) - g + 1$ ~~set $D = 0$~~, ~~also $\ell(0) =$~~

$\ell(K_X) = g$.

iii). ~~$\deg(K_X) = 2g - 2$~~ $\ell(K_X) - \ell(K_X - K_X) = \deg(K_X) - g + 1$

~~$\Rightarrow \ell(K_X) - 1 = -g + 1 + \deg(K_X)$~~

$$g - 1 = \deg(K_X) - g + 1$$

$$\deg(K_X) = 2g - 2.$$

Prop. If genus $X = 0$, X can't equiv to $\mathbb{P}^1(\mathbb{C})$

Pf: Goal: Show $\ell(pt.) \geq 2$.

$\Rightarrow \exists$ non-constant degree 1 map (one simple pole)

$$f: X \longrightarrow \mathbb{P}^1(\mathbb{C})$$

i.e. an isomorphism.

$$\ell(pt.) - \ell(K_X - pt.) = \deg(pt.) - g + 1 = 1 - 0 + 1 = 2$$

$$\Rightarrow \ell(pt.) \geq 2.$$

~~$\ell(K_X - pt.) - \ell($~~ $\quad \operatorname{rank}(D) = \ell(D) - 1$

Q: For a ~~genest~~ general curve of genus g, what is the smallest degree of a divisor of a certain rank?

(degree of divisor ⟿ degree of defining poly's, so want to be small)

want rank big so we actually get an embedding

$d(r, g)$.

Thm. (Gonality) $\quad d(1, g) = \left\lfloor \frac{g+3}{2} \right\rfloor.$

Brill - Noether Thm: (Every curve of genus g has a divisor of degree d and rank r) iff

$$g \geq (r+1)(g-d+r)$$

Two parts: 1) Show for given r, d, g that every curve has such a divisor.

2) Exhibit one curve with no such divisor of smaller degree